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On Integrals of the Hydrodynamic Equations That Correspond to Vortex Motions

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Hitherto, integrals of the hydrodynamical equations have been sought almost exclusively under the assumption that the orthogonal components of the velocity of each water particle can be set equal to the differential quotients in the corresponding directions of a certain func-

TRANSLATOR'S NOTE

Helmholtz's paper on vortex motions was first published in 1858 in the Journal fur die reine and angewandte Mathematik*. *It was the first in a series of ground-breaking papers in hydrodynamics published by Helmholtz in the decade between 1858 and 1868. In exemplary fashion it expresses his attempt at applying hydrodynamic considerations to electromagnetic phenomena, and electrodynamic models to the mathematical explication of complex hydrodynamical situations. The full force of this mode of thinking is being recognized only now when Helmholtz's basic ideas on vortex motion have found fruitful application in the investigation of high energy plasmas by researchers like Wells [this issue] and Bostick [IJFE, March 1977].*

A rough translation of Helmholtz's 1858 paper was presented by P.G. Tait in the Philosophical Magazine and Journal of Science**. *In a short postscript of his translation Tait wrote: "The above version of one of the most important recent investigations in mathematical physics was made long ago for my own use, and does not pretend to be an exact translation." The translation here is therefore the first precise rendering into English of Helmholtz's original and is presented in order to make readily available to the contemporary physicist the paper that originates the precise mathematical treatment of the concepts of vortex lines and vortex filaments that play an increasingly important role in plasma physics.*

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tion, which we shall call the velocity potential. Indeed, Lagrange has shown that this assumption is permissible whenever the motion of the water mass has been produced by and continued under the influence of forces that themselves can be represented as differential quotients of a force potential, and also that the influence of moving solids that come in contact with the fluid does not change the validity of that assumption. And, since most of the natural forces that can be well defined mathematically are representable as differential quotients of a force potential, by far the greater number of mathematically investigable cases of fluid motion fall into the category in which a velocity potential exists.

Yet, already Euler has pointed out that there are in fact cases of fluid motions in which no velocity potential exists; for instance, the rotation of a fluid about an axis with the same angular velocity for every particle. Among the forces that can produce such types of motion are magnetic forces acting upon a fluid conducting electric currents, and in particular friction, whether among the fluid particles or against fixed bodies. The influence of friction on fluids has not hitherto been mathematically definable; yet it is very great, except in the case of infinitely small oscillations, and it produces the most marked deviations between theory and reality. The difficulty of defining this effect and of finding methods for its measurement mainly consisted in the fact that no conception existed of the forms of motion that friction produces in fluids. In this regard it appeared to me to be of importance to investigate those forms of motion for which no velocity potential exists.

The following investigation will demonstrate that when there is a velocity potential, the smallest water particles have no rotational velocity, while at least a portion of the water particles is in rotation when there is no velocity potential.

By *vortex lines* I denote lines drawn through the fluid mass so that their direction at every point coincides with the direction of the momentary axis of rotation of the water particles lying on it.

By *vortex filaments* I denote portions of the fluid mass cut out from it by way of constructing corresponding vortex lines through all points of the circumference of an infinitely small surface element.

The investigation shows that if all the forces that act on the fluid have a potential: (1) no water particle that was not originally in rotation is made to rotate; (2) the water particles that at any given time belong to the same vortex line, however they may be translated, will continue to belong to the same vortex line; (3) the product of the cross section and the velocity of rotation of an infinitely thin vortex filament is constant along the entire length of the filament and retains the same value during all displacements of the filament. Hence the vortex filaments must run back into themselves in the interior of the fluid or else must end at the bounding surface of the fluid.

This last theorem enables us to determine the velocity of rotation when the form of the vortex filament at different times is given. Furthermore, the problem is solved of finding the velocities of the water particles for a given point in time if the velocities of rotation for this point in time are given; an arbitrary function, however, remains undetermined, and is to be applied to satisfy the boundary conditions.

This last problem leads to a peculiar analogy between the vortex motions of water and the electromagnetic effects of electric currents. Thus, if in a simply connected space* filled with a moving fluid there is a velocity potential, then the velocities of the water particles are equal to and in the same direction as the forces exerted on a magnetic particle in the interior of the space by a certain distribution of magnetic masses on its surface. If, on the other hand, vortex filaments exist in such a space, then the velocities of the water particles are to be set equal to the forces exerted on a magnetic particle by closed electric currents that in part flow through the vortex filaments in the interior of the mass, in part in its surface, and whose intensity is proportional to the product of the cross section of the vortex filaments and their velocity of rotation.

In the following I shall therefore frequently avail myself of the fiction of the presence of magnetic masses or of electric currents, simply in order to obtain a briefer and more vivid representation of the nature of functions that are the same kind of functions of the coordinates as the potential functions or attractive forces that attach to those masses or currents with respect to a magnetic particle.

By means of these theorems a series of forms of motion, concealed in the class of the unexamined integrals of the hydrodynamic equations, at least becomes accessible to the imagination even if the complete integration is possible only in a few of the simplest cases — as when we have one or two straight or circular vortex filaments in a mass of water that is either unlimited or partially bounded by an infinite plane.

It can be demonstrated that straight parallel vortex filaments, in a water mass limited only by planes perpendicular to the filaments, rotate about their common center of gravity, if for the determination of this point the velocity of rotation is considered analogously to the density of a mass. The position of the center of gravity remains unchanged. On the other hand, in the case of circular vortex filaments that are all perpendicular to a common axis, the center of gravity of their cross sections moves on a parallel to the axis.

1. Definition of Rotation

In a liquid capable of drop formation, at a point determined by the rectangular coordinates x, y, z, let p be the pressure at time, t; u, v, w

*I use this expression in the same sense in which Reimann (Journal fur die reine und angewandte Mathematik, Vol. 54, p. 108) speaks of simply and multiply connected surfaces.

the components of the velocity parallel to the coordinate axes; X, Y, and Z the components of external forces acting upon the unit of fluid mass; and h the density whose variations will be assumed to be vanishingly small. Then the known equations of motion for the interior points of the fluid are:

$$\left.\begin{aligned} X - \frac{1}{h}\cdot\frac{dp}{dx} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} \\ Y - \frac{1}{h}\cdot\frac{dp}{dy} &= \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} \\ Z - \frac{1}{h}\cdot\frac{dp}{dz} &= \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \\ 0 &= \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \end{aligned}\right\} \tag{1}$$

Hitherto, almost without exception, only such cases have been treated where the forces X, Y, Z, not only have a potential V so that

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}, \tag{1a}$$

but also a velocity potential Φ can be found so that

$$u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz}. \tag{1b}$$

Thereby the problem is immensely simplified, since the first three of the equations (1) give a common integral equation form which p is to be found Φ having previously been determined so as to satisfy the fourth equation, which in this case takes on the form

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0,$$

and thus coincides with the known differential equation for the potential of magnetic masses, which are external to the space for which this equation is assumed to hold. It is also known that every function φ that satisfies the above equation within a simply connected space can be

An n-fold connected space, accordingly, is one which can be cut by n-1 but no more surfaces without separating the space into two entirely separate parts. Thus a ring in this sense is a twofold connected space. The cutting surfaces all around must be bounded completely by the line in which they intersect the surface of the space.

expressed as the potential of a definite distribution of magnetic masses on the bounding surface, as I have already mentioned in the introduction.

In order that the substitution (1b) may be admissible, we must have

$$\frac{du}{dy} - \frac{dv}{dx} = 0, \quad \frac{dv}{dz} - \frac{dw}{dy} = 0, \quad \frac{dw}{dx} - \frac{dv}{dz} = 0. \tag{1c}$$

To understand the mechanical meaning of these last three conditions, we may consider the change undergone by an arbitrary infinitely small volume of water during the time dt as composed of three different motions: (1) a translation of the water particle through space; (2) an expansion or contraction of the particle parallel to three main axes of dilatation so that every rectangular parallelepiped, made out of water, whose edges are parallel to the main directions of dilatation remains rectangular, while its edges may alter their length but remain parallel to their original direction; (3) a rotation about a temporary axis of rotation of arbitrary direction, which, according to a well-known theorem, may always be considered as the resultant of three rotations about the coordinate axes.

If the conditions (1c) are fulfilled at a point whose coordinates are ($\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$), then the values of u, v, w, and of their differential quotients at the point may be put as

$$\begin{aligned}
u &= A, & \frac{du}{dx} &= a, & \frac{dw}{dy} &= \frac{dv}{dz} = \alpha, \\
v &= B, & \frac{dv}{dy} &= b, & \frac{du}{dz} &= \frac{dw}{dx} = \beta, \\
w &= C, & \frac{dw}{dz} &= c, & \frac{dv}{dx} &= \frac{du}{dy} = \gamma,
\end{aligned}$$

whence we have for a point whose coordinates x, y, z, differ by an infinitely small quantity from ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$):

$$\begin{aligned}
u &= A + a(x-\mathfrak{x}) + \gamma(y-\mathfrak{y}) + \beta(z-\mathfrak{z}), \\
v &= B + \gamma(x-\mathfrak{x}) + b(y-\mathfrak{y}) + \alpha(z-\mathfrak{z}), \\
w &= C + \beta(x-\mathfrak{x}) + \alpha(y-\mathfrak{y}) + c(z-\mathfrak{z}),
\end{aligned}$$

or, if we let

$$\begin{aligned}
\varphi = A(x-\mathfrak{x}) + B(y-\mathfrak{y}) + C(z-\mathfrak{z}) + \tfrac{1}{2}a(x-\mathfrak{x})^2 \\
+ \tfrac{1}{2}b(y-\mathfrak{y})^2 + \tfrac{1}{2}c(z-\mathfrak{z})^2 + \alpha(y-\mathfrak{y})(z-\mathfrak{z}) + \beta(x-\mathfrak{x})(z-\mathfrak{z}) \\
+ \gamma(x-\mathfrak{x})(y-\mathfrak{y}),
\end{aligned}$$

then

$$u = \frac{d\varphi}{dx}, \qquad v = \frac{d\varphi}{dy}, \qquad w = \frac{d\varphi}{dz}.$$

It is known that by an appropriate choice of differently oriented coordinates *x, y, z* with origin at ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) the expression for ϕ can be brought into the form

$$\varphi = A_1 x_1 + B_1 y_1 + C_1 z_1 + \tfrac{1}{2} a_1 x_1^2 + \tfrac{1}{2} b_1 y_1^2 + \tfrac{1}{2} c_1 z_1^2,$$

where the components *u, v, w,* of the velocity with respect to the new coordinate axes have the values

$$u_1 = A_1 + a_1 x_1, \quad v_1 = B_1 + b_1 y_1, \quad w_1 = C_1 + c_1 z_1.$$

Velocity u_1 parallel to the x_1 axis is thus the same for all water particles for which x_1 has the same value, so that water particles that at the beginning of time *dt* are in a plane parallel to the $y_1 z_1$ plane will at the end of *dt* also lie in such a plane. The same holds for the $x_1 y_1$ *and the* $x_1 z_1$ plane. Thus if we imagine a parallelepiped bounded by three planes, parallel and infinitely close to the just mentioned coordinate planes, then the enclosed water particles after the passage of time *dt* will still form a rectangular parallelepiped whose surfaces are parallel to the same coordinate planes. Thus, the entire motion of such an infinitely small parallelepiped is, given assumptions (lc), composed of only a translation in space and a dilatation or contraction of its edges, but does not involve any rotation.

Let us return to the first coordinate system of *x, y, z,* and suppose that aside from the motions of the infinitely small water mass surrounding point ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) in existence so far, there exist additional rotational motions around axes through point ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) and parallel to the *x, y,* and *z* axes, whose angular velocities are ξ, η, ζ, Then the velocity components parallel to the coordinate axes of *x, y, z,* contributed by these motions are:

$$\begin{array}{ccc} 0, & (z-\mathfrak{z})\,\xi, & -(y-\mathfrak{y})\,\xi, \\ -(z-\mathfrak{z})\,\eta, & 0, & (x-\mathfrak{x})\,\eta \\ (y-\mathfrak{y})\,\zeta, & -(x-\mathfrak{x})\,\zeta, & 0. \end{array}$$

Thus the velocities of the particle with coordinates x, y, z now become:

$$u = A + a(x - \mathfrak{x}) + (\gamma + \zeta)(y - \mathfrak{y}) + (\beta - \eta)(z - \mathfrak{z}),$$
$$v = B + (\gamma - \zeta)(x - \mathfrak{x}) + b(\mathfrak{y} - y) + (\alpha + \xi)(z - \mathfrak{z}),$$
$$w = C + (\beta + \eta)(x - \mathfrak{x}) + (\alpha - \xi)(y - \mathfrak{y}) + c(z - \mathfrak{z}).$$

From these follows by differentiation:

$$\left\{\begin{aligned} \frac{dv}{dz} - \frac{dw}{d\gamma} &= 2\,\xi, \\ \frac{dw}{dx} - \frac{du}{dz} &= 2\,\eta, \\ \frac{du}{dy} - \frac{dv}{dx} &= 2\,\zeta. \end{aligned}\right. \tag{2}$$

The magnitudes on the left, therefore, which according to equations (1c) must be equal to zero, if a velocity potential is to exist, are equal to twice the rotational velocities of the water particles around the three coordinate axes. The existence of a velocity potential excludes the existence of rotational motions of the water particles.

As a further characteristic property of fluid motion with a velocity potential, we shall adduce here that no such motion can exist in a simply connected space, S, *which is completely filled with a fluid enclosed by* completely rigid walls. For if n denotes the normal of the surface of such a sphere, pointing to the interior, then the velocity component perpendicular to the wall $d\varphi/dn$ must be zero everywhere. Then, according to a well-known theorem by Green*:

$$\iiint\left[\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2 + \left(\frac{d\varphi}{dz}\right)^2\right]dx\,dy\,dz = -\int \varphi\,\frac{d\varphi}{dn}\,d\omega,$$

where the integral on the left is to be extended over the entire space S, the integral on the right over the entire surface of S, a surface element of which is denoted by $d\omega$. Now, if $d\varphi/dn$ is equal to zero over the entire surface, then the integral on the left, too, must be equal to zero, which can be the case only if throughout the whole space S:

$$\frac{d\varphi}{dx} = \frac{d\varphi}{dy} = \frac{d\varphi}{dz} = 0,$$

thus, if no motion of the water takes place at all. Every motion of a bounded fluid mass in a simply connected space, when a velocity potential exists, therefore necessarily implies a motion of the fluid surface. If this surface motion, that is $d\varphi/dn$, is given in full, then as a result of this the entire motion of the enclosed fluid mass, too, is

* The previously noted theorem, which is not valid for multiply connected spaces.

uniquely determined. For if there were two functions $\varphi_,$ and $\varphi_{,,}$, which in the interior of space S were to simultaneously satisfy equation:

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=0$$

and at the surface were to satisfy the condition:

$$\frac{d\varphi}{dn}=\psi$$

where ψ denotes the values of $d\varphi/dn$ determined by the given surface motion, then the function $(\varphi_,-\varphi_{,,})$, too, would satisfy the first condition in the interior of S, while at the surface we would have:

$$\frac{d(\varphi_,-\varphi_{,,})}{dn}=0$$

which, as just demonstrated, for the whole interior of S would imply:

$$\frac{d(\varphi_,-\varphi_{,,})}{dx}=\frac{d(\varphi_,-\varphi_{,,})}{dy}=\frac{d(\varphi_,-\varphi_{,,})}{dz}=0.$$

To both functions, therefore, exactly the same velocities also would correspond in the whole interior of S.

Thus rotations of the water particles and motions on a closed curve can occur in simply connected and entirely closed spaces only if no velocity potential exists. Therefore, in general, we may call motions without velocity potential vortex motions.

2. CONSTANCY OF VORTEX MOTION

First, we shall determine the variations of the rotational velocities ξ, η and ζ during the motion, when the only active forces are those that have a force potential. I first note in general that, if ψ is a function of x, y, z, and t and increases by $\partial\psi$ while the latter four magnitudes increase by ∂x, ∂y, ∂z, and ∂t, we have:
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$$\partial\psi=\frac{d\psi}{dt}\partial t+\frac{d\psi}{dx}\partial x+\frac{d\psi}{dy}\partial y+\frac{d\psi}{dz}\partial z.$$

If we now want to determine the change of ψt during the time interval ∂t for a water particle that remains constant, then we have to give to the magnitudes ∂x, ∂y, and ∂z the same values they have for the moving water particle, that is:

$$\partial x = u\partial t, \quad \partial y = v\partial t, \quad \partial z = w\partial t,$$

and we obtain:

$$\frac{\partial \psi}{\partial t} = \frac{d\psi}{dt} + u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}.$$

In what follows I shall employ the symbol $\partial\psi/\partial t$ only in the sense that $(\partial\psi/\partial t)\, dt$ denotes the change of ψ during the time dt for the specific water particle, whose coordinates at the beginning of the time interval dt were x, y, and z.

If we elminate the magnitudes p by differentiation from the first of the equations (1) and simultaneously introduce the notation of equations (2), regarding equations (1a) as satisfiable for the forces X, Y, Z, we obtain the following three equations:

$$\left.\begin{aligned} \frac{\partial \xi}{\partial t} &= \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}, \\ \frac{\partial \eta}{\partial t} &= \xi\frac{dv}{dx} + \eta\frac{dv}{dy} + \zeta\frac{dv}{dz}, \\ \frac{\partial \zeta}{\partial t} &= \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz}, \end{aligned}\right\} \tag{3}$$

or equivalently:

$$\left.\begin{aligned} \frac{\partial \xi}{\partial t} &= \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx}, \\ \frac{\partial \eta}{\partial t} &= \xi\frac{du}{dy} + \eta\frac{dv}{dy} + \zeta\frac{dw}{dy}, \\ \frac{\partial \zeta}{\partial t} &= \xi\frac{du}{dz} + \eta\frac{dv}{dz} + \zeta\frac{dw}{dz}. \end{aligned}\right\} \tag{3a}$$

If in a water particle ξ, η, and ζ simultaneously are equal to zero, then also:

$$\frac{\partial \xi}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \zeta}{\partial t} = 0.$$

Hence, those water particles that do not already possess rotational motion do not attain such motion as time goes on.

As is well known, rotations can be composed according to the method of the parallelogram for forces. If ξ, η, ζ are the rotational velocities around the coordinate axes, then the rotational velocity q around the momentary axis of rotation is:

$$q = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

and the cosines of the angles of this axis with the coordinates are:

$$\xi / q, \ \eta / q \text{ und } \zeta / q.$$

If we now take in the direction of this momentary axis of rotation the infinitely small **portion** $q\varepsilon$, then the projections of this portion onto the three coordinate axis are $\varepsilon\xi$, $\varepsilon\eta$, $\varepsilon\zeta$. While at **point** x, y, z the components of the velocity are u, v, w, at the other endpoint of $q\varepsilon$ they are:

$$u_1 = u + \varepsilon\xi \frac{du}{dx} + \varepsilon\eta \frac{du}{dy} + \varepsilon\zeta \frac{du}{dz},$$
$$v_1 = v + \varepsilon\xi \frac{dv}{dx} + \varepsilon\eta \frac{dv}{dy} + \varepsilon\zeta \frac{dv}{dz},$$
$$w_1 = w + \varepsilon\xi \frac{dw}{dx} + \varepsilon\eta \frac{dw}{ny} + \varepsilon\zeta \frac{dw}{dz}.$$

At the end of time dt, therefore, the projections of the distance between the two particles, which at the beginning of dt limited the portion $q\varepsilon$, have attained values that, taking into account equations (3), may be written as follows:

$$\varepsilon\xi + (u_1 - u)\, dt = \varepsilon \left(\xi + \frac{\partial \xi}{\partial t} dt \right),$$
$$\varepsilon\eta + (v_1 - v)\, dt = \varepsilon \left(\eta + \frac{\partial \eta}{\partial t} dt \right),$$
$$\varepsilon\zeta + (w_1 - w)\, dt = \varepsilon \left(\zeta + \frac{\partial \zeta}{\partial t} dt \right).$$

The left-handed sides of these equations give the projections of the new position of the connecting line $q\varepsilon$, the right-hand sides the projection of the new velocity of rotation, multiplied by the constant factor ϵ ; it follows from these equations that the connecting line between the two water particles, which at the beginning of time dt bounded the portion $q\varepsilon$ of the momentary axis of rotation, also after the lapse of time dt still coincides with the now-altered axis of rotation.

If we call *vortex line* a line whose direction coincides everywhere with the momentary axis of rotation of the water particles situated there, as we defined above, then the just-found theorem can be enunciated as follows: *Each vortex line remains continually composed of the same water particles, while it swims forward with these water particles in the fluid.*

The rectangular components of the velocity of rotation increase in the same proportion as the projections of the portion εq of the axis of rotation; from this it follows that *the magnitude of the resulting velocity of rotation in a specific water particle varies in the same proportion as the distance of this water particle from its neighbor in the axis of rotation.*

If we imagine that vortex lines are drawn through every point of the circumference of an infinitely small surface, then as a result of this a filament of infinitely small cross section is separated out from the fluid, which we shall call *vortex filament*. The volume of the portion of such a filament is bounded by two specific water particles, which, according to the just-proved theorems, is always filled by the same water particles, must remain constant during the motion, and its cross section, therefore, must vary inversely to its length. Hence the just-stated theorem also may be enunciated as follows: *The product of the velocity of rotation and the cross section in a portion of a vortex filament containing the same water particles remains constant during the motion of the filament.*

From equations (2) it follows immediately that:

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 .$$

And, further, from this that:

$$\iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \right) dx\, dy\, dz = 0 ,$$

where the integral may be extended over an entirely arbitrary portion S of the water mass. Through integration by parts it follows:

$$\iint \xi\, dy\, dz + \iint \eta\, dx\, dz + \iint \zeta\, dx\, dy = 0 ,$$

where the integrals are to be extended over the entire surface of the space S. Calling an element of this surface $d\omega$ and α, β, γ the three angles made with the coordinate axes by the normal $d\omega$, drawn outwards, we have:

$$dy\, dz = \cos\alpha\, d\omega , \quad dx\, dz = \cos\beta\, d\omega , \quad dx\, dy = \cos\gamma\, d\omega .$$

Hence:

$$\iint (\xi \cos\alpha + \eta \cos\beta + \zeta \cos\gamma)\, d\omega = 0 ,$$

or if σ is the resulting velocity of rotation, and ϑ the angle between its axis and the normal,

$$\int\int \sigma \cos \vartheta \,.\, d\omega = 0,$$

the integral extending over the entire surface of S.

Now let S be a portion of a vortex filament, bounded by two infinitely small planes $\omega_{,}$ and $\omega_{,,}$ perpendicular to the axis of the filament. Then $\cos \vartheta$ is equal to 1 at one of these planes, equal to -1 at the other, and equal to 0 at the rest of the surface of the filament. Consequently, if $\sigma_{,}$ and $\sigma_{,,}$ are the rotational velocities in $\omega_{,}$ and $\omega_{,,}$ the last equation reduces to:

$$\sigma_{,} \omega_{,} = \sigma_{,,} \omega_{,,},$$

from which it follows that *the product of the velocity of rotation and the cross section is constant throughout the entire length of a given vortex filament.* That it does not change as a result of the motion of the filament has been proven previously.

It also follows from this that a vortex filament can never end within the fluid, but must either return ring-shaped into itself within the fluid or reach to the boundaries of the fluid; for if a vortex filament ended anywhere within the fluid, a closed surface could be constructed for which the integral $\int \sigma \cos \vartheta$ would not have the value zero.

3. SPATIAL INTEGRATION

If the motion of the vortex filaments in the fluid can be determined, the stated theorems also will enable us to determine the magnitudes ξ, η, and ζ completely. We shall now consider the problem of finding the velocities $u, v,$ and w from the magnitudes ξ, η, and ζ.

Thus, let there be given within a water mass that fills the space S the values of ξ, η, and ζ, which three magnitudes satisfy the condition that:

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0. \tag{2a}$$

We want to find $u, v,$ and w, so that within the entire space S they satisfy the conditions that:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{1}$$

$$\left.\begin{aligned} \frac{dv}{dz} - \frac{dw}{dy} &= 2\xi, \\ \frac{dw}{dx} - \frac{du}{dz} &= 2\eta, \\ \frac{du}{dy} - \frac{dv}{dx} &= 2\zeta. \end{aligned}\right\} \tag{2}$$

In addition there are the conditions determined by the particular nature of a given problem for the boundary of the space S.

For a given distribution of ξ, η, ζ we now may have vortex lines that within the space S are closed and return into themselves, as well as some that reach the boundary of S and there break off. If the latter is the case, one can continue these vortex lines, either on the surface of S or outside of S, and close them so that they return into themselves, so that a larger space S then comes into existence that contains only closed vortex lines, and for the entire surface of which ξ, η, ζ and their resultant σ are each equal to zero, or at least:

$$\xi \cos \alpha + \eta \cos \beta + \zeta \cos \gamma = \sigma \cos \vartheta = 0. \tag{2b}$$

As previously α, β, γ denote the angles between the normal of the portion of the surface of S under consideration and the coordinate axes, ϑ the angle between the normal and the resulting axis of rotation.

We obtain values of u, v, w, that satisfy the equations $(1)_4$ and (2) if we put:

$$\left.\begin{aligned} u &= \frac{dP}{dx} + \frac{dN}{dy} - \frac{dM}{dz}, \\ v &= \frac{dP}{dy} + \frac{dL}{dz} - \frac{dN}{dx}, \\ w &= \frac{dP}{dz} + \frac{dM}{dx} - \frac{dL}{dy}, \end{aligned}\right\} \tag{4}$$

and determine the magnitudes L, M, N, P by means of the conditions that within the space S:

$$\left.\begin{aligned} \frac{d^2L}{dx^2} + \frac{d^2L}{dy^2} + \frac{d^2L}{dz^2} &= 2\xi, \\ \frac{d^2M}{dx^2} + \frac{d^2M}{dy^2} + \frac{d^2M}{dz^2} &= 2\eta, \\ \frac{d^2N}{dx^2} + \frac{d^2N}{dy^2} + \frac{d^2N}{dz^2} &= 2\zeta, \\ \frac{d^2P}{dx^2} + \frac{d^2P}{dy^2} + \frac{d^2P}{dz^2} &= 0. \end{aligned}\right\} \tag{5}$$

The method of integrating these latter equations is known. L, M, N, are the potential functions of imaginary magnetic masses distributed through the space S with the densities$-\xi/2\pi$, $-\eta/2\pi$, and $-\zeta/2\pi$ the potential function of masses which lie outside the space S. If we denote by r the distance of a point, whose coordinates are a, b, c, from the point x, y, z, and by ξ_a, η_a, ζ_a the values of ξ, η, ζ at the point a, b, c, then we have:

$$\left.\begin{aligned} L &= -\frac{1}{2\pi}\iiint \frac{\xi_a}{r}\, da\, db\, dc \\ M &= -\frac{1}{2\pi}\iiint \frac{\eta_a}{r}\, da\, db\, dc \\ N &= -\frac{1}{2\pi}\iiint \frac{\zeta_a}{r}\, da\, db\, dc, \end{aligned}\right\} \quad (5a)$$

the integration extending over the space S and:

$$P = \iiint \frac{k}{r}\, da\, db\, dc,$$

where k is an arbitrary function of a, b, c, and the integration is to be extended over the exterior space enclosing S. The arbitrary function k must be taken so as to satisfy the boundary conditions, a problem whose difficulty is similar to that concerning electric and magnetic distribution.

That the values of u, v, and w, given in (4), satisfy condition (1) is proved by differentiation, taking into account the fourth of equations (5).

We further find by differentiation of equations (4), taking into account the first three of equations (5), that:

$$\begin{aligned} \frac{dv}{dz} - \frac{dw}{dy} &= 2\xi - \frac{d}{dx}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right] \\ \frac{dw}{dx} - \frac{du}{dz} &= 2\eta - \frac{d}{dy}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right] \\ \frac{du}{dy} = \frac{dv}{dx} &= 2\zeta - \frac{d}{dz}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right]. \end{aligned}$$

Equations (2) are thus also satisfied, if it can be shown that in the entire space S :

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = 0. \quad (5b)$$

That this is the case follows from equations (5a):

$$\frac{dL}{dx} = +\frac{1}{2\pi}\iiint \frac{\xi_a(x-a)}{r^3}\, da\, db\, dc,$$

or, after integration by parts:

$$\frac{dL}{dx} = \frac{1}{2\pi}\iint \frac{\xi_a}{r} db\, dc - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\xi_a}{da} da\, db\, dc,$$

$$\frac{dM}{dy} = \frac{1}{2\pi}\iint \frac{\eta_a}{r} da\, dc - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\eta_a}{db} da\, db\, dc,$$

$$\frac{dN}{dz} = \frac{1}{2\pi}\iint \frac{\zeta_a}{r} da\, db - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\zeta_a}{dc} da\, db\, dc.$$

Adding these three equations, and again calling the surface element of S $d\omega$, we obtain:

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = \frac{1}{2\pi}\int (\xi_a \cos\alpha + \eta_a \cos\beta + \zeta_a \cos\gamma)\frac{1}{r} d\omega$$
$$- \frac{1}{2\pi}\iiint \frac{1}{r}\left(\frac{d\xi_a}{da} + \frac{d\eta_a}{db} + \frac{d\zeta_a}{dc}\right) da\, db\, dc.$$

Since, however, throughout the interior of the space:

$$\frac{d\xi_a}{da} + \frac{d\eta_a}{db} + \frac{d\zeta_a}{dc} = 0, \qquad (2a)$$

and on its entire surface:

$$\xi_a \cos\alpha + \eta_a \cos\beta + \zeta_a \cos\gamma = 0, \qquad (2b)$$

therefore both integrals are equal to zero and equation (5b) as well as equations (2) are satisfied. Equations (4) and (5) or (5a) are thus indeed integrals of equations $(1)_4$ and (2).

The analogy mentioned in the introduction between the distance-actions of vortex filaments and the electromagnetic distance-actions of current-conducting wires, which provides a very good means of clearly demonstrating the form of vortex motions, is deducible from these theorems.

If we substitute in equation (4) the values of L, M, N from equations (5a), and denote by Δu, Δv, Δw those infinitely small parts of $u, v,$

and w which in the integrals result from the element da, db, dc, also their resultant by Δp, we have:

$$\Delta u = \frac{1}{2\pi}\frac{(y-b)\,\zeta_a - (z-c)\,\eta_a}{r^3}\,da\,db\,dc\,,$$

$$\Delta v = \frac{1}{2\pi}\frac{(z-c)\,\xi_a - (x-a)\,\zeta_a}{r^3}\,da\,db\,dc\,,$$

$$\Delta w = \frac{1}{2\pi}\frac{(x-a)\,\eta_a - (y-b)\,\xi_a}{r^3}\,da\,db\,dc.$$

From these equations it follows that:

$$\Delta u\,(x-a) + \Delta v\,(y-b) + \Delta w\,(z-c) = 0,$$

that is, the resultant Δp of Δu, Δv and Δw is at right angles to r. Further:

$$\xi_a \Delta u + \eta_a \Delta v + \zeta_a \Delta w = 0\,,$$

that is, the same resultant Δp is also at right angles to the resulting axis of rotation at a, b, c. Finally:

$$\Delta p = \sqrt{\Delta u^2 + \Delta v^2 + \Delta w^2} = \frac{da\,db\,dc}{2\pi r^2}\,\sigma \sin \nu\,,$$

where σ is the resultant of ξ_a, η_a, ζ_a and ν the angle it makes with r, which is determined by means of the equation:

$$\sigma r \cos \nu = (x-a)\,\xi_a + (y-b)\,\eta_a + (z-c)\,\zeta_a\,.$$

Each rotating water particle a *thus determines in every other particle* b *of the same water mass a velocity whose direction is perpendicular to the plane through the axis of rotation of* a *and particle* b. *The magnitude of this velocity is directly proportional to the volume of* a, *its velocity of rotation, and the sine of the angle between the line* ab *and the axis of rotation, and inversely proportional to the square of the distance between both particles.*

Exactly the same law holds for the force that would be exerted by an electric current at *a*, parallel to the axis of rotation, on a magnetic particle at *b*.

The mathematical similarity of these two classes of natural phenomena rests upon this, that in the case of water vortices, for those parts of the water mass that have no rotation, a velocity potential exists that satisfies the equation:

$$\frac{d^2 \varphi}{dx^2} + \frac{d^2 \varphi}{dy^2} + \frac{d^2 \varphi}{dz^2} = 0$$

which equation holds everywhere except within the vortex filaments. If however, we consider the vortex filaments as always closed either within or outside of the water mass, then the space for which the differential equation for φ is valid is multiply connected, for it remains connected, if we imagine surfaces of separation through it, each of which is completely bounded by a vortex filament. In such multiply connected spaces a function φ that satisfies the above differential equation can become multivalued; and it must become multivalued if it is to represent currents returning into themselves; for, since the velocities of the water mass outside the vortex filaments are proportional to the differential quotients of φ, following the motion of the water one must progress to ever increasing values of φ. Therefore, if the current returns into itself, and if following it one finally arrives at the place where one had been previously, one finds for this place a second higher value of φ. Since the same procedure can be carried out infinitely often, there must exist infinitely many different values of φ for each point of such a multiply connected space that differ by the same differences, much as in the case of the different values of Arc tan [x/y], which is such a multivalued function satisfying the above differential equation.

Such also is the case with the electromagnetic effects of a closed electric current. This acts at a distance just as a specific distribution of magnetic masses on a surface bounded by the conductor. Outside the current, therefore, the forces it exerts on a magnetic particle can be considered as the differential quotients of a potential function V that satisfies the equation:

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

But in this case, too, the space that surrounds the closed conductor and in which this equation holds, is multiply connected, and V is multivalued.

Thus, in the case of vortex motions of water as in the case of electromagnetic effects, the velocities or forces outside the space traversed by vortex filaments or electric currents depend upon multivalued potential functions, which incidentally satisfy the general differential equation for magnetic potential functions, while inside the space penetrated by vortex filaments or electric currents instead of the potential functions, which do not exist here, different common functions of the kind appearing in equations (4), (5), and (5a) arise. On the other hand, in the case of simply streaming water motions and magnetic forces we are dealing with single-valued potential functions just as in the cases of gravitation, forces of electric attraction, and constant electric and thermal currents.

Those integrals of the hydrodynamical equations for which a single-valued velocity potential exists, we may call integrals of the first class. On the other hand, those where there is rotation of some of the water particles and consequently a multivalued velocity potential in the nonrotating water particles, we may call integrals of the second class. It may occur that in the latter case only those portions of the space are to be treated in the problem that contain no rotating water particles; for instance, in the case of motions of water in ring-shaped vessels, where a vortex filament may be supposed to lie along the axis of the vessel, and where the problem, therefore, still belongs to those that can be solved by the assumption of a velocity potential.

In the hydrodynamical integrals of the first class, the velocities of the water particles are in the same direction as and proportional to the forces that a specific distribution of magnetic masses outside the fluid would exert on a magnetic particle at the place of the water particle.

In the hydrodynamic integrals of the second class the velocities of the water particles are in the same direction as and proportional to the forces that would act on a magnetic particle and that would be produced by closed electric currents flowing through the vortex filaments with a density proportional to the velocity of rotation of these filaments, combined with magnetic masses outside the fluid. The electric currents inside the fluid would have to move with their respective vortex filaments and retain constant intensity. The assumed distribution of magnetic masses outside the fluid or on its surface must be taken so that the boundary conditions are satisfied. Each magnetic mass also, as we know, can be replaced by electric currents. Thus, instead of using for the values of u, v, w, the potential function P of an external mass k, we obtain a solution of the same generality if we give ξ, η, and ζ outside of, or even just at the surface of, the fluid arbitrary values such that only closed current filaments are generated; and then the integration in equations (5a) must be extended over the whole space in which ξ, η, and ζ are different from zero.

4. VORTEX SURFACES AND ENERGY OF VORTEX FILAMENTS

In hydrodynamic integrals of the first class, it is sufficient, as I have shown above, to know the motion of the surface. By this the motion in the interior of the fluid is entirely determined. In integrals of the second class, on the other hand, it is necessary, in addition, to determine the motion of the vortex filament in the interior of the fluid taking into account their mutual interaction and respecting the boundary conditions, which makes the problem much more complicated. Even this problem, however, can be solved for certain simple cases — specifically, when rotation of the water particles occurs only in certain surfaces or lines and the form of these surfaces or lines remains unchanged during the motion.

The properties of the surfaces bounded by an infinitely thin layer of rotating water particles can be deduced easily from equations (5a). If ξ, η, and ζ differ from zero only in an infinitely thin layer, their potential functions L, M, and N, according to known theorems, will have the same values on both sides of the layer, while their differential quotients, taken in the direction of the normal of the layer, will be different. If we assume the coordinate axes so placed that at the point of the vortex surface under consideration the z-axis corresponds to the normal of the surface and the x-axis to the axis of rotation of the water particles in the surface so that at this point $\eta = \zeta = 0$, then the potentials M and N as well as their differential quotients will have the same values on both sides of the layer. The same holds for L and dL/dx and dL/dy, while dL/dz will have two different values, whose difference is equal to $z\,\xi\varepsilon$, if ε denotes the thickness of the layer. Consequently equations (4), for u and w, yield the same values on both sides of the vortex surface, for v, however, values which differ by $2\xi\varepsilon$. Hence, that component of the velocity that is a tangent to the vortex surface and at right angles to the vortex lines differs in value on both sides of the surface. Within the layer of rotating water particles, the component of the velocity under consideration must be thought of as uniformly increasing from the value on one side of the surface to that on the other. For if ξ is constant here through the entire thickness of the layer, and α represents a proper fraction, v^1 the value of v on one, v_1 on the other side, v_α its value in the layer itself at a distance $\alpha\epsilon$ from the first side, then we saw that $v'-v_1=2\xi\epsilon$, because between both sides there is a layer of thickness ϵ and of intensity of rotation ξ. For the same reason we must have $v'-v_\alpha=2\xi\epsilon\alpha=\alpha(v'-v_1)$, which expresses the above theorem. Since we must consider the rotating water particles as themselves moved, and the change of their distribution on the surface depends on their motion, we must assign to them a mean velocity of flow along the surface for the entire thickness of the layer, which corresponds to the arithmetical mean of the velocities on both sides of the layer.

Such a vortex surface would be produced, for example, when two previously separate moving masses of fluid come into contact with each other. At the surface of contact the velocities perpendicular to the surface necessarily would have to become equal. However, the velocities tangential to the surface will be different, in general, in the two fluid masses. The surface of contact thus would have the properties of a vortex surface.

On the other hand, isolated vortex filaments cannot, in general, be supposed infinitely thin, since the velocities at opposite sides of the filament then would attain infinitely great and opposite values, and the velocity of the filament itself would become indefinite. To nonetheless obtain certain general conclusions about the motion of very thin filaments of arbitrary cross section, we will make use of the principle of the conservation of vis viva.

Thus, before we proceed to specific examples, we will first form the equation for the vis viva K of the moving mass of water:

$$K = \tfrac{1}{2} h \iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz.$$

We now from equations (4) substitute in this integral:

$$u^2 = u\left(\frac{dP}{dx} + \frac{dN}{dy} - \frac{dM}{dz}\right),$$
$$v^2 = v\left(\frac{dP}{dy} + \frac{dL}{dz} - \frac{dN}{dx}\right),$$
$$w^2 = w\left(\frac{dP}{dz} + \frac{dM}{dx} - \frac{dL}{dy}\right)$$

and integrate by parts, denoting by $\cos\alpha$, $\cos\beta$, $\operatorname{Cos}\gamma$, and $\cos\vartheta$ the angles, which the inwardly directed normal of the element $d\omega$ of the water mass makes with the coordinate axes and with the resultant velocity q; we thus obtain, taking into account equations (2) and (1) :

$$K = -\frac{h}{2}\int d\omega\,[\,Pq\cos\vartheta + L(v\cos\gamma - w\cos\beta) \qquad (6\text{ a})$$
$$+ M(w\cos\alpha - u\cos\gamma) + N(u\cos\beta - v\cos\alpha)]$$
$$- h\iiint (L\xi + M\eta + N\zeta)\, dx\, dy\, dz.$$

The value of dK/dt is obtained from equations (1) by multiplying the first by u, the second by v, the third by w, and adding:

$$h\left(u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt}\right) = -\left(u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz}\right)$$
$$+ h\left(u\frac{dV}{dx} + v\frac{dV}{dy} + w\frac{dV}{dz}\right) - \frac{h}{2}\left(u\frac{d(q^2)}{dx} + v\frac{d(q^2)}{dy} + w\frac{d(q^2)}{dz}\right).$$

if both sides are multiplied by $dxdydz$ and then we integrate over the entire extent of the water mass, noticing that because of $(1)_4$:

$$\iiint\left(u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}\right) dx\, dy\, dz = -\int \psi\, q\cos\vartheta\, d\omega,$$

if ψ denotes a continuous and single valued function in the interior of the water mass, we obtain:

$$\frac{dK}{dt} = \int d\omega\,(p - hU + \tfrac{1}{2}hq^2)\, q\cos\vartheta. \qquad (6\text{ b})$$

If the water mass is entirely enclosed within rigid walls, $q\cos\vartheta$ must be zero at all points on the surface. Hence also $dK/dt=0$, that is, K=constant.

If we consider this rigid wall as being at an infinite distance from the origin of the coordinates, and all existing vortex filaments at a finite distance, then the potential functions *L, M, N,* whose masses ξ, η, ζ are each in sum equal to zero, will at an infinite distance $\mathfrak{R}$ decrease proportional to $\mathfrak{R}^{-2}$, and the velocities, their differential quotients, will decrease as $\mathfrak{R}^{-3}$,; but the surface element dw if it is always to correspond to the same solid angle at the coordinate origin, will increase proportional to $\mathfrak{R}^{2}$. The first integral in the expression for K[equation (6a)], which is extended over the surface of the water mass, will decrease as $\mathfrak{R}^{-3}$, and, therefore, will vanish for an infinite value of $\mathfrak{R}$. Thus the value of K reduces to:

$$K = -h \iiint (L\xi + M\eta + N\zeta)\, dx\, dy\, dz \qquad (6c)$$

and this value does not alter during the motion.

5. STRAIGHT PARALLEL VORTEX FILAMENTS

First we shall consider the case where only straight vortex filaments parallel to the z-axis exist, whether in an infinitely extended mass of water or in a similar mass limited by two infinite planes perpendicular to the filaments, which amounts to the same thing. All motions then occur in planes perpendicular to the z-axis and are exactly the same in all such planes.

We therefore put:

$$w = \frac{du}{dz} = \frac{dv}{dz} = \frac{dp}{dz} = \frac{dV}{dz} = 0.$$

Then equations (2) reduce to:

$$\xi = 0, \quad \eta = 0, \quad 2\zeta = \frac{du}{dy} - \frac{dv}{dx},$$

equations (3) to:

$$\frac{\delta\zeta}{\partial t} = 0.$$

The vortex filaments thus retain constant rotational velocity as well as constant cross section.

Equations (4) reduce to:

$$u = \frac{dN}{dy}, \qquad v = -\frac{dN}{dx}, \quad \frac{d^2N}{dx^2} + \frac{d^2N}{dy^2} = 2\zeta.$$

In accordance with the remark at the end of section 3, we have here put $P=0$. The equation of the stream lines is therefore $N=$constant.

N is in this case the potential function of infinitely long lines; it is itself infinitely great, but its differential quotients are finite. If a and b are the coordinates of a vortex filament, whose cross section is *da db*, then we have:

$$-v = \frac{dN}{dx} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{x-a}{r^2}, \qquad u = \frac{dN}{dy} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{y-b}{r^2}.$$

From this it follows that the resultant velocity q is perpendicular to r, which is the perpendicular to the vortex filament, and that:

$$q = \frac{\zeta\, da\, db}{\pi r}.$$

If in a water mass infinitely extended in the directions of x and y we have several vortex filaments, whose coordinates are x , y , x , y , and so forth, and we denote the product of the rotational velocity and the cross section of each of these by m , m and so forth, then, forming the sums:

$$U = m_1 u_1 + m_2 u_2 + m_3 u_3 \text{ etc.},$$
$$V = m_1 v_1 + m_2 v_2 + m_3 v_3 \text{ etc.},$$

these will each be equal to zero, since the portion of the sum V that arises from the effect of the second vortex filament on the first is canceled by the effect of the first on the second. For both are:

$$m_1 \cdot \frac{m_2}{\pi} \frac{x_1 - x_2}{r^2} \text{ und } m_2 \cdot \frac{m_1}{\pi} \frac{x_2 - x_1}{r^2}$$

and so for all the others in both sums. Now U is the velocity of the center of gravity of the masses $\mathbf{m}_1$, $\mathbf{m}_2$ and so forth, in the direction of $\mathbf{x}$ multiplied by the sum of these masses; so of V in the direction of y. Both velocities are thus zero, unless the sum of the masses is zero, in which case there is no center of gravity. Thus the center of gravity of the vortex filaments remains unchanged during their motions about one another; and since this theorem holds for any arbitrary distribution of vortex filaments we also may apply it to isolated vortex filaments of infinitely small cross section.

From this we derive the following consequences:

1. In case of a single rectilinear vortex filament of infinitely small cross section in a water mass infinite in all directions perpendicular to the vortex filament, the motion of the water particles at a finite distance from it depends only on the product $\zeta\, da\, db = m$ of the rotational velocity and the magnitude of its cross section, not on the form of its cross section. The particles of the water mass rotate about it with tangential velocity $m/\pi r$, where r denotes the distance from the center of gravity of the vortex filament. The position of the center of gravity itself, the rotational velocity, the magnitude of the cross section, and thus also the magnitude m remain unchanged, even if the form of the infinitely small cross section may alter.

2. In case of two rectilinear vortex filaments of infinitely small cross section in an unlimited water mass, each will cause the other to move in a direction perpendicular to the line connecting them. The length of the connecting line is not changed as a result of this. Thus, both will rotate about their common center of gravity at constant distances from it. If the rotational velocity in both vortex filaments is of the same direction, that is, of the same sign, then their center of gravity must lie between them. If it is of opposite direction, that is, of different signs, then their center of gravity lies in the prolongation of the line connecting them. And if the product of the rotational velocity and the cross sections is the same for both, but of opposite sign, so that the center of gravity would lie at an infinite distance, they both travel forward with equal velocity and in the same direction perpendicular to the line connecting them.

To the latter case may also be referred that in which a vortex filament of infinitely small cross section moves next to an infinitely extended plane that is parallel to it. The boundary condition for the motion of the water touching the plane, that is, that it must be parallel to the plane, is satisfied by imagining that beyond the plane there is a second vortex filament, the mirror image of the first. From this it follows that the vortex filament in the water mass travels forward parallel to the plane in the direction in which the water particles between it and plane move, and with one-fourth of the velocity possessed by the water particles at the foot of a perpendicular from the vortex filament onto the plane.

For rectilinear vortex filaments the assumption of an infinitely small cross section leads to no inadmissible consequences, since no individual filament exerts a propelling action upon itself, but is propelled only by the influence of the other filaments present. It is different for curved filaments.

6. CIRCULAR VORTEX FILAMENTS

Assume that in an infinitely extended water mass there exist only two circular vortex filaments, whose planes are perpendicular to the z-axis and whose centers lie in this axis so that all around them everything is symmetrical. Let the coordinates be changed by putting:

$$\begin{aligned} x &= \chi \cos \varepsilon, & a &= g \cos e, \\ y &= \chi \sin \varepsilon, & b &= g \sin e, \\ z &= z, & c &= c. \end{aligned}$$

According to our assumption the velocity of rotation σ is a function only of χ and z or of g and c, and the axis of rotation is everywhere perpendicular to χ (or g) and the z-axis. Thus the rectangular components of the rotation at the point with coordinates $g, e,$ and c are:

$$\xi = -\sigma \sin e, \quad \eta = \sigma \cos e, \quad \zeta = 0.$$

In equations (5a) we obtain:

$$r^2 = (z-c)^2 + \chi^2 + g^2 - 2\chi g \cos(\varepsilon - e),$$
$$L = \frac{1}{2\pi} \iiint \frac{\sigma \sin e}{r} g\, dg\, de\, dc,$$
$$M = -\frac{1}{2\pi} \iiint \frac{\sigma \cos e}{r} g\, dg\, de\, dc,$$
$$N = 0.$$

By multiplying with $\cos \varepsilon$ and $\sin \varepsilon$, and adding, one obtains from the equations for L and M:

$$L \sin\varepsilon - M \cos\varepsilon = -\frac{1}{2\pi} \iiint \frac{\sigma \cos(\varepsilon - e)}{r} g\, dg\, d(\varepsilon - e)\, dc,$$
$$L \cos\varepsilon + M \sin\varepsilon = \frac{1}{2\pi} \iiint \frac{\sigma \sin(\varepsilon - e)}{r} g\, dg\, d(\varepsilon - e)\, dc.$$

In both integrals the angles e and ε occur only in the form of $(\varepsilon - e)$, and this magnitude, therefore can be regarded as the variable under the integral. In the second integral, the elements in which $(\varepsilon - e) = \mathfrak{e}$ are canceled by those in which $(\varepsilon - e) = 2\pi - \mathfrak{e}$, and it is, therefore, equal to zero. If we put:

$$\psi = \frac{1}{2\pi} \iiint \frac{\sigma \cos e \,.\, g\, dg\, de\, dc}{\sqrt{(z-c)^2 + \chi^2 + g^2 - 2g\chi \cos e}}, \qquad (7)$$

we therefore obtain:

$$M \cos\varepsilon - L \sin\varepsilon = \psi,$$
$$M \sin\varepsilon + L \cos\varepsilon = 0,$$

or:

$$L = -\psi \sin\varepsilon, \qquad M = \psi \cos\varepsilon. \qquad (7a)$$

Calling τ the velocity in the direction of the radius χ, and taking into account that because of the symmetrical position of the vortex rings about the axis the velocity in the direction of the circumference must be equal to zero, we have:

$$u = \tau \cos\varepsilon, \qquad v = \tau \sin\varepsilon,$$

and according to equations (4):

$$u = \frac{dM}{dz}, \qquad v = \frac{dL}{dz}, \qquad w = \frac{dM}{dx} - \frac{dL}{dy}.$$

From this it follows:

$$\tau = -\frac{d\psi}{dz}, \qquad w = \frac{d\psi}{d\chi} + \frac{\psi}{\chi},$$

or

$$\tau\chi = -\frac{d(\psi\chi)}{dz}, \qquad w\chi = \frac{d(\psi\chi)}{d\chi}. \qquad (7b)$$

The equation of the streamlines is therefore:

$$\psi\chi = \text{Const.}$$

If first we carry out the integration indicated in the value of ψ for a vortex filament of infinitely small cross section, putting and labeling the part of ψ obtained from this by ψ_{m_1}, then: $\sigma dg\, dc = m_1$,

$$\psi_{m_1} = \frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\left\{\frac{2}{\varkappa}(F - E) - \varkappa F\right\},$$

$$\varkappa^2 = \frac{4g\chi}{(g+\chi)^2 + (z-c)^2},$$

where F and E are the complete elliptic integrals of the first and second kind for the modulus

Putting, for the sake of brevity,

$$U = \frac{2}{\varkappa}(F - E) - \varkappa F,$$

where, therefore, U is a function of $\varkappa$, we have:

$$\tau\chi = \frac{m_1}{\pi}\sqrt{g\chi}\,\frac{dU}{d\varkappa}\cdot\varkappa\cdot\frac{z-c}{(g+\chi)^2 + (z-c)^2}.$$

Now, if at the point given by χ and z there exists a second vortex filament m, and if by τ_1 we denote the velocity in the direction of g that it imparts to the vortex filament m_1, then we obtain the latter if in the expression for τ we put τ_1 g χ c z m. instead of τ χ g z c m_1

In this $\varkappa$ and U remain unchanged and we obtain:

$$m\tau\chi + m_1\tau_1 g = 0. \qquad (8)$$

If now we determine the value w of the velocity parallel to the axis, which is produced by the vortex filament m_1, whose coordinates are g and c, we find:

$$w\chi = \frac{1}{2}\frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\,U + \frac{m_1}{\pi}\sqrt{g\chi}\,\frac{dU}{d\varkappa}\cdot\frac{\varkappa}{2\chi}\cdot\frac{(z-c)^2 + g^2 - \chi^2}{(g+\chi)^2 + (z-c)^2}.$$

If we call w_1 the velocity parallel to the z-axis produced by the vortex ring m, whose coordinates are z and χ, at the position of m_1, then we need only make again the previously indicated interchange of the relevant coordinates and masses. Then one finds that:

$$2mw\chi^2 + 2m_1 w_1 g^2 - m\tau\chi z - m_1\tau_1 g c = \frac{2mm_1}{\pi}\sqrt{g\chi}\,U. \qquad (8a)$$

Sums similar to (8) and (8a) can be formed for an arbitrarily large number of vortex rings. For the nth of these the product $\sigma\, dg\, dc$ is denoted by m_n, the components of the velocity, that is imparted to it by the other vortex filaments by τ_n and w_n, where, however, we omit for the time being the velocities that each vortex ring can impart to itself. I also call the radius of the ring ϱ_n and λ, its distance from a plane perpendicular to the axis, which two magnitudes coincide in direction with χ and z, but as belonging to a particular vortex ring are functions of the time and not independent variables like χ and z. Finally, let the value of ψ, so far as it arises from the other vortex rings, be ψ_n. From (8) and (8a), by writing out the corresponding equations for every single pair of vortex rings and adding all of them, we obtain:

$$\Sigma[m_n\varrho_n\tau_n] = 0,$$

$$\Sigma[2m_n w_n\varrho_n^2 - m_n\tau_n\varrho_n\lambda_n] = \Sigma[m_n\varrho_n\psi_n].$$

As long as in these sums we still have a finite number of separate and infinitely thin vortex rings, we can consider w, τ, and ψ only as those parts of these magnitudes that are produced by the presence of the other rings. If, however, we suppose the space to be continuously filled with an infinitely large number of such rings, ψ is the potential function of a continuous mass, w and τ are the differential quotients of this potential function; and it is known that for such a function as well as for its differential quotients the parts of the function due to the presence of mass in an infinitely small neighborhood around the point for which the function is determined are infinitely small in comparison with those due to finite masses at a finite distance.*

If, therefore, we change the sums into integrals, then we can consider w, τ, and ψ the total values of these magnitudes at the point in question, and put:

$$w = \frac{d\lambda}{dt}, \qquad \tau = \frac{d\varrho}{dt}$$

For this purpose we replace the magnitude m with the product $\sigma\, d\varrho\, d\lambda$.

$$\iint \sigma\varrho \frac{d\varrho}{dt}\, d\varrho\, d\lambda = 0, \tag{9}$$

$$2\iint \sigma\varrho^2 \frac{d\lambda}{dt}\, d\varrho\, d\lambda - \iint \sigma\varrho\lambda \frac{d\varrho}{dt}\, d\varrho\, d\lambda = \iint \sigma\varrho\psi\, d\varrho\, d\lambda. \tag{9a}$$

Since according to section 2 the product $\sigma\, d\varrho\, d\lambda$ is constant with respect to time, equation (9) can be integrated with respect to t and we obtain:

$$\tfrac{1}{2}\iint \sigma\varrho^2\, d\varrho\, d\lambda = \text{Const.}$$

If we consider the space to be divided by a plane that passes through the z-axis and therefore cuts all existing vortex rings; if we further consider σ as the density of a layer of mass, and call $\mathfrak{M}$ the entire mass in this layer of the plane, that is:

$$\mathfrak{M} = \iint \sigma\, d\varrho\, d\lambda,$$

and R^2 the mean value of ϱ^2 for all the elements of mass, then:

$$\iint \sigma\varrho \cdot \varrho\, d\varrho\, d\lambda = \mathfrak{M}R^2,$$

and since this integral and the value of $\mathfrak{M}$ are constant with respect to time, it follows that R, too, also remains unchanged during the forward motion.

Therefore, if there exists in the unlimited fluid mass only one circular vortex filament of infinitely small cross section, then its radius remains unchanged.

*Cf Gauss in *Resultate des magnetischen Vereins*, 1839, p. 7.

According to equation (6c) the magnitude of the vis viva is in our case:

$$K = -h \iiint (L\xi + M\eta)\, da\, db\, dc$$
$$= -h \iiint \psi\sigma . \varrho\, d\varrho\, d\lambda\, d\varepsilon$$
$$= -2\pi h \iint \psi\, \sigma . \varrho\, d_{\cdot\, l}\, l\lambda.$$

It, too, is constant with respect to time.

We further note that since $\sigma\, d\varrho\, d\lambda.$ is constant with respect to time,

$$\frac{d}{dt}\iint \sigma\varrho^2\lambda\, d\varrho\, d\lambda = 2\iint \sigma\varrho\lambda\, \frac{d\varrho}{dt}\, d\varrho\, d\lambda + \iint \sigma\varrho^2\, \frac{d\lambda}{dt}\, d\lambda\, d\varrho,$$

and thus equation (9a), if by l we denote the value of λ for the center of gravity of the cross section of the vortex filament and multiply (9) by it and add, becomes:

$$2\,\frac{d}{dt}\iint \sigma\varrho^2\lambda\, d\varrho\, d\lambda + 5\iint \sigma\varrho\,(l-\lambda)\,\frac{d\varrho}{dt}\, d\varrho\, d\lambda = -\,\frac{K}{2\pi h}. \qquad (9b)$$

If the cross section of the vortex filament is infinitely small, and ε is an infinitely small magnitude of the same order as $l-\lambda$ and the other linear dimensions of the cross section, while $\sigma\, d\varrho\, d\lambda$ is finite, then ψ as well as K are of the same order of infinitely large quantities as log ε . For very small values of the distances v from the vortex ring we have:

$$v = \sqrt{(g-\chi)^2 + (z-c)^2},$$
$$\varkappa^2 = 1 - \frac{v^2}{4g^2},$$
$$\psi_{m_1} = \frac{m_1}{\pi}\log\left(\frac{\sqrt{1-\varkappa^2}}{4}\right) = \frac{m_1}{\pi}\log\frac{v}{8g}.$$

In the value of K, ψ is further multiplied by ϱ or g. If g is finite and v of the same order as ϵ then K is of the order of log ε. Only if g is infinitely large of the order of $1/\varepsilon$, K becomes infinitely large as $(1/\varepsilon)\log\varepsilon$. The the circle becomes a straight line. But, on the other hand, $d\varrho/dt$ which is equal to $d\psi/dz$, becomes of the order $1/\varepsilon$, the second integral therefore is finite and for finite ϱ vanishingly small compared to K. In this case we may put the constant l in place of λ, and obtain:

$$2\,\frac{d(\mathfrak{M}R^2 l)}{dt} = -\,\frac{K}{2\pi h}$$

or:

$$2\,\mathfrak{M}R^2 l = C - \frac{K}{2\pi h}\,t.$$

Since $\mathfrak{M}$ and R are constant, l can vary only proportional to time. If $\mathfrak{M}$ is positive, the motion of the water particles on the outer side of the ring is in the direction of positive z, on the inner side in the direction of negative z; K, h and R from their nature always positive.

From this, therefore, it follows that in case of a circular vortex filament of very small cross section in an infinitely extended water mass the center of gravity of the cross section has motion parallel to the axis of the vortex ring of approximately constant and very high velocity, which is directed to the same side to which the water flows through the ring. Infinitely thin vortex filaments of finite radius would attain infinitely great translational velocities. If, however, the radius of the vortex filament is infinitely great of order $1/\varepsilon$, then R^2 becomes infinitely great in comparison with K, and l becomes constant. The vortex filament, which now has changed into a straight line, becomes stationary, as we have already found earlier for rectilinear vortex filaments. Now we can also see in general how two ringlike vortex filaments with the same axis will mutually affect each other, since each in addition to its own motion also follows the motion of the water particles produced by the other. If they have the same direction of rotation, they both travel in the same direction; the one in front will widen and then travel more slowly, the pursuing one will contract and travel faster, till finally at velocities not too different, it will catch up with the other and go through it. Then this same game will be repeated with the other one so that, in turn, the rings will pass one through the other.

If the vortex filaments have equal radii and equal and opposite velocities of rotation, they will approach each other and widen one another, so that finally when they have come very close to each other their velocity of approach becomes smaller and smaller; the widening, on the other hand, occurs with increasing velocity. If the two vortex filaments are entirely symmetrical, the velocity of the water particles midway between the two and parallel to the axis is equal to zero. Thus one might imagine a rigid wall inserted here without disturbing the motion, and so obtain the case of a vortex ring that runs up against a rigid wall.

Finally, I remark that these motions of circular vortex rings are easily studied in nature by rapidly drawing for a short space along the surface of a fluid a half-immersed circular disc or the approximately semicircular point of a spoon and quickly withdrawing it. Half vortex rings then remain in the fluid whose axis lies in the free surface. The free surface thus forms a limiting plane of the water mass placed through the axis, but as a result of which there is no essential change in the motions. The vortex rings travel on, widen when they come to a wall, and are widened or contracted by other vortex rings exactly as we have deduced it from the theory.