SCIENTIFIC CLASSICS

Lectures on The Integral Calculus

Johann Bernoulli

(Translated by William A. Ferguson, Jr.)

The first English translation of selections from Bernoulli's groundbreaking work identifying the curve formed by a hanging chain suspended at both ends.

Johann Bernoulli (1667-1748)

I. Lecture Thirty-Six
On Catenaries

The importance of the problem of the catenary in Geometry can be seen from the three solutions in the *Acta* of Leipzig of last year (1691), and especially from the remarks that the renowned Leibniz[1] makes there. The first to consider this curve, which is formed by a free-hanging string, or better, by a thin inelastic chain, was Galileo. He, however, did not fathom its nature; on the contrary, he asserted that it is a parabola, which it certainly is not. Joachim Jungius discovered that it is not a parabola, as Leibniz remarked, through calculation and his many experiments. However, he did not indicate the correct curve for the catenary. The solution to this important problem therefore remained for our time. We present it here together with the calculation, which was not appended to the solution in the *Acta.*

There are actually two kinds of catenaries: the common, which is formed by a string or a chain of uniform thickness, or is of uniform weight at all points, and the uncommon, which is formed by a string of non-uniform thickness, which therefore is not of uniform weight at all points, and certainly not uniform in relation to the ordinates of any given curve.

Before we set about the solution, we make the following assumptions, which can easily be proven from Statics.

1. The string, rope, or chain, or whatever the curve consists of, will be assumed to be flexible and inelastic at all of its points, that is, it undergoes no stretching as a result of its weight.
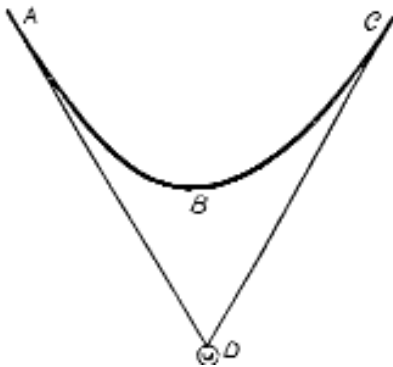

Figure 1

2. If the catenary *ABC* [Figure 1] is held fixed at any two points *A* and *C,* then the necessary forces at points *A* and *C,* are the same as those which support a weight *D,* that is equal to the weight of the chain *ABC* and is located at the meeting point of two weightless strings *AD* and *CD,* that are tangent to the curve at points *A* and *C.* The reason for this is clear: Because the weight of the chain *ABC* exerts its action at *A* and *C* in one direction [at each point], namely, in the directions of the tangents *AD* and *CD,* and the pull of the same or equal weight *D* at *A* and *C* likewise goes in the directions of *AD* and *CD.* Therefore the necessary forces at points *A* and *C* must also in both cases be the same. Accordingly, one obtains the nec-

essary force at the lowest point *B,* when one seeks the force that the weight *E* [Figure 2] exerts at the same point, when it is held by two weightless strings, one of which is tangent to the curve at *B,* and therefore is horizontal, while the other is tangent to the curve at point *A.*
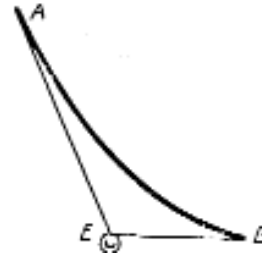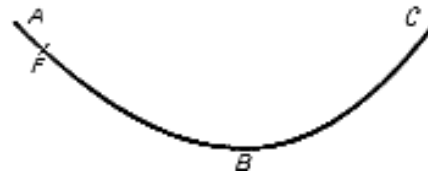

Figure 2

Figure 3

3. When a chain fastened at points *A* and *C* is then fastened at any other point *F* [Figure 3], so that one could remove the portion *AF,* the curve represented by the remaining piece of chain *FBC* does not change, that is, the remaining points will stay in the same position as before the fastening [at *F*].

This needs no proof, because Reason advises it and experience lays it daily before our eyes.

4. If we retain the previous assumptions, then before and after the fastening [at *F*], the same (that is, the original) force must obtain at particular positions on the curve, or, what amounts to the same thing, a point will be pulled with the same force after the fastening [at *F*] as before it. This is nothing but a corollary of the preceding number. Consequently, as one lengthens or shortens the chain *BFA,* that is, wherever one chooses the fastening point *F,* the force at the lowest position *B* neither increases nor decreases, but always remains the same.

5. The weight *P* [Figure 4], which is held by any two arbitrarily situated strings *AB* and *CB,* exerts its forces on the points *A* and *C* in such a relation, that the necessary force at *A* is to the necessary force at *C* (after drawing vertical line *BG*), as the sine of angle *CBG* is to the sine of angle *ABG,* and the force of the weight *P* is to the force at *C* as the sine of the whole angle *ABC* is to the sine of the opposite angle *ABG.* This is proven in every theory of Statics.[2]

With these assumptions, we find the common catenary curve in the following manner. Let *BAa* be the desired curve [Figure 5]; *B,* its deepest point; the axis or the vertical through *B, BG;* the tangent at the deepest point, which will be horizontal, *BE;* and let *AE* be the tangent at any other point *A.*

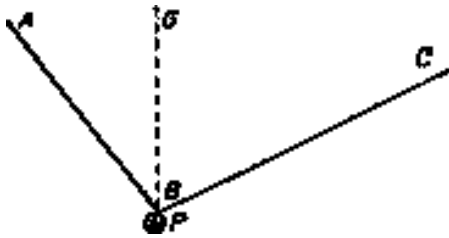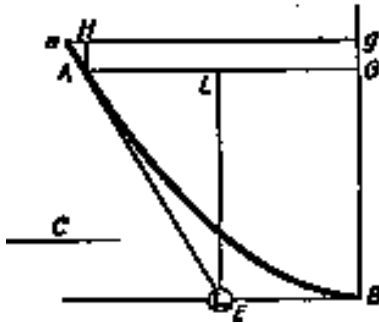Draw the ordinate *AG* and the parallel *EL* to the axis. Let

Figure 4

Figure 5

$$BG = x, \quad GA = y, \quad Gg = dx, \quad Ha = dy,$$

and the weight of the chain, or, since it is of uniform thickness, the length of the curve *BA = s.* Since at point *B,* an ever constant force will be required (by assumption 4), whether the chain be lengthened or shortened, that force, or the segment *C = a* expressing it, will therefore be a constant.[3] Imagine now that the weight of the chain *AB* is concentrated at and hangs at the meeting point *E* of the tangent strings *AE, BE;* then (by assumption 2) the same force is required at point *B* to hold the weight *E* as was required to hold the chain *BA.* But the weight *E* (by assumption 5) is to the force at *B,* as the sine of the angle *AEB,* or as the sine of its complementary angle *EAL* is to the sine of angle *AEL,* that is, as *EL* is to *AL.* Wherever on the curve one chooses the fixed point *A* (the curve always remains the same, by assumption 3), the weight of the chain *AB* is to the force at *B* [which force equals the constant *a*], as *EL* is to *AL,* that is,

$$s : a = EL : AL = AH : Ha = dx : dy$$

and if one inverts,

$$dy : dx = a : s. \quad [4]$$

Hence it follows that the catenary *BA* is the same as that curve whose construction and nature we have given above, by the method of inverse tangents [provided here as Part II, below], where we first converted the proportion $dy : dx = a : s$ to the following:

$$dy = \frac{a\,dx}{\sqrt{2ax + x^2}},$$

at which point the curve was constructed through the rectification of the parabola as well as through the quadrature of the hyperbola.

II. Part of Lecture Twelve on Inverse Tangents

IV. To find the nature of the curve so created that $DC : BC = E : AD$ [Figure 6].
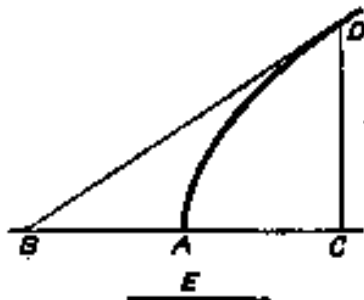
Figure 6

Let *AC = x, CD = y, AD = s* [and the given constant segment *E = a*]. By assumption,

$$\frac{dy}{dx} = \frac{a}{s}, \quad \text{therefore } dy = \frac{a\,dx}{s}.$$

However, to be able to eliminate the letter *s* (which is always necessary in the determination of curves), one must proceed thus:

$$dy^2 = \frac{a^2 dx^2}{s^2},$$

therefore,

$$ds^2 = dx^2 + dy^2 = \frac{s^2 dx^2 + a^2 dx^2}{s^2}$$

and

$$ds = \frac{dx\sqrt{s^2 + a^2}}{s},$$

therefore

$$dx = \frac{s\,ds}{\sqrt{s^2 + a^2}}$$

and the integral thereof,

$$x = \sqrt{s^2 + a^2}.$$

From this is obtained

$$s = \sqrt{x^2 - a^2}$$

and

$$ds = \frac{xdx}{\sqrt{x^2 - a^2}} = \sqrt{dx^2 + dy^2}.$$

If the equation is simplified, one obtains

$$x^2 dy^2 - a^2 dy^2 = a^2 dx^2$$

and finally

$$dy = \frac{adx}{\sqrt{x^2 - a^2}}.$$

The same result is achieved otherwise and more easily in the following manner.

Because $s = adx/dy$, then[5]

$$ds = \sqrt{dx^2 + dy^2} = \frac{ad^2x}{dy}$$

and hence

$$dy = \frac{ad^2x}{\sqrt{dx^2 + dy^2}}.$$

To be able to take the integral on both sides, both sides are multiplied by dx. Then one obtains

$$dxdy = \frac{adxd^2x}{\sqrt{dx^2 + dy^2}}.$$

If the integral be taken, the result is

$$xdy = a\sqrt{dx^2 + dy^2},$$

and after simplifying the equation,

$$dy = \frac{adx}{\sqrt{x^2 - a^2}},$$

as before.

Now we come to the construction of this curve. It should be noted, first of all, that because $x = \sqrt{s^2 + a^2}$ and hence $x > s$, the invariant origin of x lies beyond the vertex B, and indeed at the distance E, since if $s = o$, then $x = a$ necessarily. Hence if we wish to place the origin of x at the vertex itself, we must set $x = x + a$. Then the equation $dy = adx/\sqrt{x^2 - a^2}$ is transformed into the following:

$$dy = \frac{adx}{\sqrt{2ax + x^2}},$$

which we will now construct in a three-fold way.

Multiply the equation by a. Then

$$ady = \frac{a^2 dx}{\sqrt{2ax + x^2}}.$$
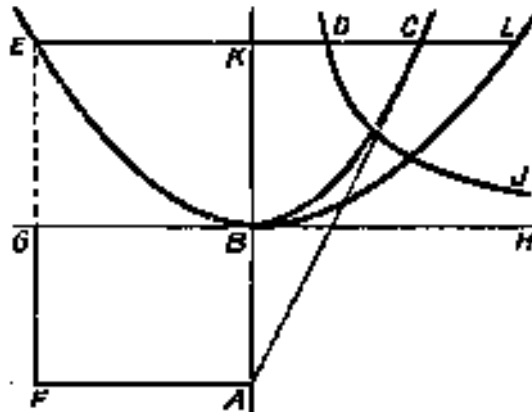
Figure 7

Now draw the normals AK, GH, which will intersect at B [Figure 7]; take $BA = a$ and draw with vertex B and midpoint A the equilateral hyperbola BC. Construct further a curve DJ, such that BA is everywhere the mean proportional between KC and KD, that is, that

$$KD = \frac{a^2}{\sqrt{2ax + x^2}}.$$

Next, draw the parallel AF and make the area of the rectangle AG equal to that of $HBKDJ$. If we now extend DK and FG, their intersection E will lie on the desired curve.[6]

The curve admits of another and easier construction in the following manner. Draw the line AC and make the area of rectangle AG equal to double the hyperbolic area ABC. Then after the extension of CK and FG, the point E again will lie on the same desired curve.[7, 8]

It may be constructed in yet another way, by means of the rectification of a parabolic curve, in the following manner. Because $dy = adx/\sqrt{2ax + x^2}$, therefore

$$dy + \frac{adx + xdx}{\sqrt{2ax + x^2}} \quad \text{(differential of } EK + KC = EC\text{)}$$
$$= \frac{2adx + xdx}{\sqrt{2ax + x^2}} = \frac{dx\sqrt{2a + x}}{\sqrt{x}}.$$

Hence one must find a certain curve BL, whose differential is $dx\sqrt{2a + x}/\sqrt{x}$. Then BL itself will be equal to EC. But one finds this curve thus: From

$$\frac{2adx^2 + xdx^2}{x},$$

subtract dx^2. Then $2adx^2/x$ remains. Hence

$$\frac{dx\sqrt{2a}}{\sqrt{x}} = \text{differential of } KL,$$

and

$$\int \frac{dx\sqrt{2a}}{\sqrt{x}}, \text{ that is, } \sqrt{8ax},$$

will therefore be equal to *KL* itself. The curve *BL* is consequently a parabola of parameter 8*AB.* If this *BL* be laid out as a straight line and abutted to point *C* as ordinate, the other endpoint *E* again will lie on the desired curve *BE.*[9,10]

Corollary. The length of the curve *BE* is equal to the hyperbolic ordinate *KC.* If the length of *BE* is designated by *s,* then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{a^2 dx^2}{2ax + x^2}} = dx\sqrt{\frac{a^2 + 2ax + x^2}{2ax + x^2}}$$

$$= \frac{adx + xdx}{\sqrt{2ax + x^2}} = \text{differential of } KC.$$

III. Lecture Thirty-Seven
Continuation of the Same Subject: On Catenaries

So that the correctness of our solution be made even clearer, we will test whether it agrees with the solution of Mr. Leibniz. His construction of the catenary is of the following nature.
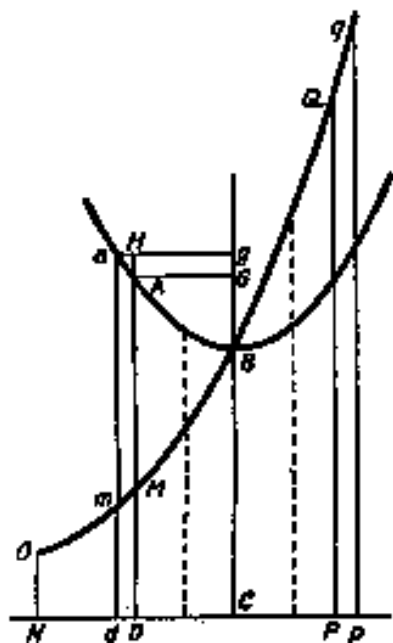

Figure 8

Let *NCP* be an unbounded horizontal straight line [Figure 8], and above it let the logarithmic curve *OMBQ* be drawn,[11] whose subtangent therefore is everywhere the same. Choose the ordinate *CB,* which is equal to the subtangent, and take on both sides of it arbitrary and equal segments *CD, CP.* Now make *DA* equal to the semi-sum of the ordinates *DM, PQ.* Then, he asserts that the point *A* lies on a catenary.[12]

In order to test, whether this curve is the same as the one we have indicated, we must see whether the nature of the curve *BA* is expressed by the same differential equation. Let then *CB* or the subtangent = *a, GB = x, GA = CD = y, DM = z, Gg = dx, Dd = Ha = dy.* Then, by the nature of the logarithmic curve, *zdy = adz,* thus

$$dz = \frac{zdy}{a}.$$

Since by construction, *CD = CP,* then *DM : CB = CB : PQ,*

therefore $PQ = a^2/z$ and $\frac{1}{2}DM + \frac{1}{2}PQ$, that is, *DA* by construction,

$$= \frac{a^2 + z^2}{2z} = CB + BG = a + x,$$

therefore

$$z^2 = 2az + 2xz - a^2.$$

When solved, this equation yields,

$$z = a + x + \sqrt{2ax + x^2},$$

therefore

$$dz = dx + \frac{(a + x)dx}{\sqrt{2ax + x^2}}.$$

If one substitutes this value of *z* in the earlier equation $dz = zdy/a$, the result is

$$dx + \frac{(a + x)dx}{\sqrt{2ax + x^2}} = \frac{ady + xdy + dy \cdot \sqrt{2ax + x^2}}{a}$$

or

$$\frac{adx \cdot \sqrt{2ax + x^2} + a^2dx + axdx}{\sqrt{2ax + x^2}}$$

$$= ady + xdy + dy \cdot \sqrt{2ax + x^2}.$$

Dividing both sides by $a + x + \sqrt{2ax + x^2}$, one obtains

$$\frac{adx}{\sqrt{2ax + x^2}} = dy.$$

Because this equation is the same as the one that we have found, it follows that the curve *BA* is also our catenary, and that the Leibnizian construction, as different as it may be from the one we have given above, indeed produces no other line.

It remains to add the most important properties of the simple catenary, and that with calculation and proof, which was not done in the *Acta.* We will use the figure that appears in the *Acta* [Figure 9]. There, *EBF* is the catenary, *B* its deepest point, *BA* the axis, *BG* the equilateral hyperbola that can be termed generative, and *BH* the parabola, through whose rectification the catenary line *EBF* is constructed.
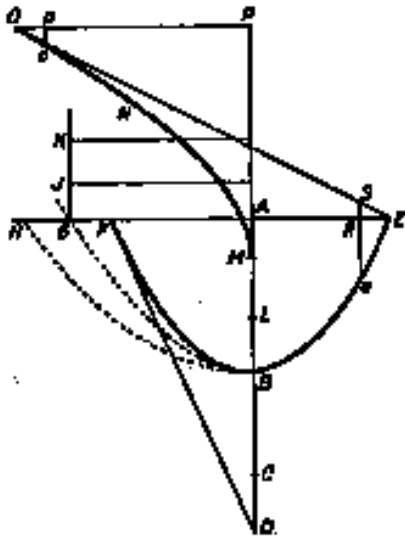
Figure 9

1. Draw the tangent *FD;* then *AF* : *AD* = *BC* : (*BF,* the curve), because

$$AF : AD = dy : dx.$$

However we found by calculation that

$$dy : dx = a : s.$$

Therefore, the statement is correct.

2. *AE* or *AF* is equal to the parabolic curve *BH,* minus the segment *AG.* This is clear, because by construction, *EG* was taken equal to *BH.*

3. The length of the curve *BE* or *BF* is equal to the segment *AG;* that is, the portions of the catenary, if one lays them out upon the axis as ordinates, form an equilateral hyperbola.[13] That is a remarkable property of this curve. We have proven this by the method of inverse tangents (see page 31).

4. The area of the catenary region *BAE* or *BAF* is equal to the rectangle of *BA* and *AF,* minus the rectangle of *CB* and *FG.* Since of course

$$dy = \frac{adx}{\sqrt{2ax + x^2}}$$

therefore

$$xdy = \frac{axdx}{\sqrt{2ax + x^2}} = \frac{(ax + a^2)dx}{\sqrt{2ax + x^2}} - \frac{a^2dx}{\sqrt{2ax + x^2}}$$

$$= \frac{(ax + a^2)dx}{\sqrt{2ax + x^2}} - ady.$$

Therefore the integral of *xdy,* that is, the complement of the area *BAF,* is equal to the integral of the latter, and therefore equal to

$$a\sqrt{2ax + x^2} - ay = CB \cdot AG - CB \cdot AF = CB \cdot FG,$$

and consequently the area *BAF* itself $= BA \cdot AF - CB \cdot FG.$

5. The length of the curve *MNO,* the involute of which

forms the catenary *BE,* is the third proportional of *CB* and *AG.* To discover that this is so, one would first find the length of the unwinding tangent line *EO;* above, in the article on the development of curves [in Lecture Sixteen, not translated here], we have shown that in general, for all curves, it is equal to

$$\frac{(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{-d^2ydx}.$$

For the curve in question then,

$$dx^2 + dy^2 = ds^2 = \frac{(a^2 + 2ax + x^2)dx^2}{2ax + x^2},$$

and, since

$$dy = \frac{adx}{\sqrt{2ax + x^2}},$$

hence

$$d^2y = -\frac{(a^2 + ax)dx^2}{(2ax + x^2)\sqrt{2ax + x^2}}.$$

From this is obtained, for the whole expression,

$$\frac{(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{-d^2ydx} = EO = \frac{a^2 + 2ax + x^2}{a}.$$

From this, subtract that which the assumption *x* = *o* yields; then what remains is

$$\frac{2ax + x^2}{a} = \text{ the length of curve } MNO.$$

Therefore *a* or *CB* is to $\sqrt{2ax + x^2}$ or *AG,* as *AG* is to *MNO.*

6. The unwinding tangent *EO* is the third proportional of *CB* and *CA.* For, since

$$EO = \frac{a^2 + 2ax + x^2}{a},$$

a or *CB* is to *a* + *x* or *CA* as *CA* is to *EO.*

7. The line *BM,* which extends to the beginning of the curve *MNO,* is equal to *CB.* Since of course *x* = *o,* the unwinding tangent *EO,* which is now *BM,* becomes equal to *a* = *CB.*

8. *MP* is twice *BA.* Because

$$MNO = \frac{2ax + x^2}{a},$$

the differential will be

$$Oo = \frac{(2a + 2x)dx}{a}.$$

But the triangle *Oop* is similar to triangle *ESR* and hence also to triangle *eER;* consequently *Ee* : *ER* = *Oo* : *po,* that is,

$$\frac{adx + xdx}{\sqrt{2ax + x^2}} : \frac{adx}{\sqrt{2ax + x^2}} = \frac{2adx + 2xdx}{a} : po,$$

therefore $2dx = po$ and, if integrated, $2x = PM$.[14]

9. The area of the rectangle of *CB* and *PO* is twice that of the hyperbolic area *ABG*. Since of course *Ee* : *Oo* = *eR* : *pO,* that is,

$$\frac{adx + xdx}{\sqrt{2ax + x^2}} : \frac{2adx + 2xdx}{a} = dx : pO,$$

therefore

$$pO = \frac{2dx \cdot \sqrt{2ax + x^2}}{a},$$

thus

$$CB \cdot pO = 2dx\sqrt{2ax + x^2},$$

and, when integrated,

$$CB \cdot PO = \text{double the hyperbolic area } ABG.$$

10. The segment *CP* is bisected by the point *A.* Since of course *MP* = 2*x*, the result is that *BP* = 2*x* + *a*, *BP* − *BA* or *AP* = *x* + *a* = *CA.*

11. The length of the curve *EB* is to the length of the curve *MNO* as the segment *CB* is to segment *AG*. That is because *EB,* that is, $\sqrt{2ax + x^2}$, is to *MNO* or

$$\frac{2ax + x^2}{a}$$

as *a* is to $\sqrt{2ax + x^2}$, and therefore as *CB* is to *AG.*

12. If the two rectangles *AJ, AK* are placed on *AG* (*AJ* is formed from the half transverse axis *CB* and the segment *FG*, and the area of *AK* is equal to that of the hyperbolic area *BGA*); and, from the vertex *B* along the axis, a segment *BL* equal to width *KJ* is drawn; then point *L* will be the center of gravity of the catenary *EBF.* This will be proven in another location.

13. If one imagines infinitely many curves drawn from *E* to *F* that are equal in length to the catenary *EBF* and lays them out as straight lines, and constructs individual vertical lengths at the individual points of the particular segment equal to the respective distances from the line *EF,* then of all areas formed in this manner, the greatest is that formed by the catenary.[15]

This will be proven with the help of the axiom that the center of gravity descends as far as it can.
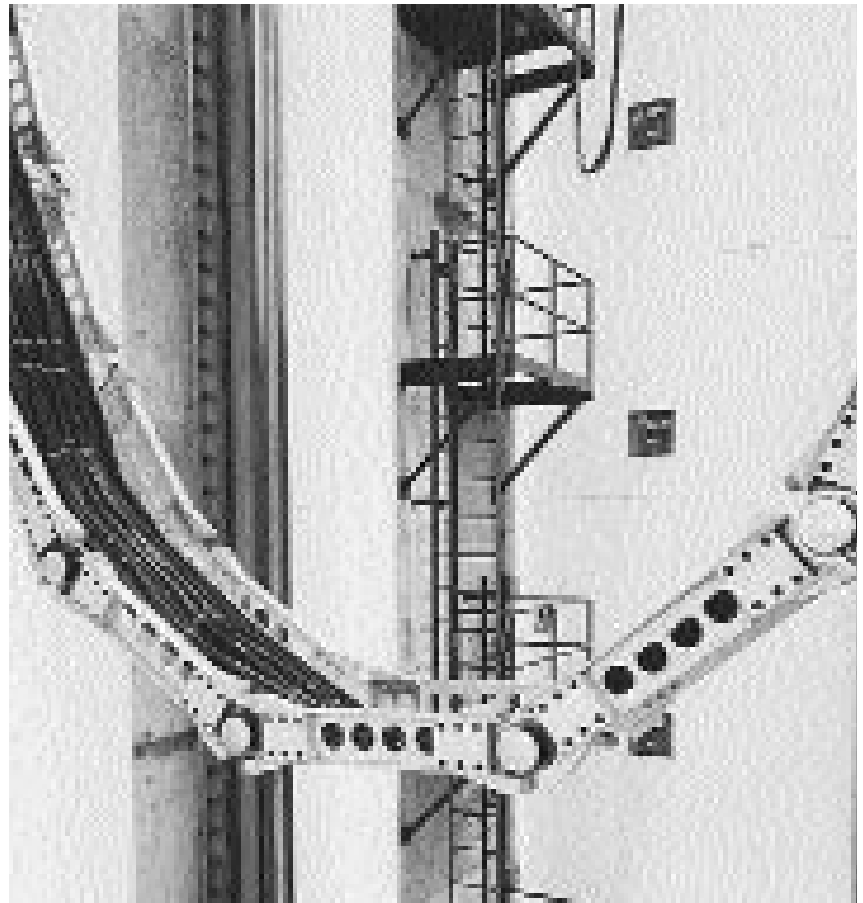
Two huge chain catenaries used in the testing of aircraft at NASA's Vertical Motion Flight Simulation Laboratory.

Notes

1. In Galileo's *Dialogues Concerning Two New Sciences* (1638), the catenary is discussed and its form compared to a parabola. Joachim Jungius (1669) showed that the curve is not a parabola, but he was not really able to define it. Jakob Bernoulli proposed the problem anew in May 1690 in the *Acta Eruditorum,* and then it was solved by Huygens, Leibniz, and Johann Bernoulli. [Kowalewski]

2. Let the force acting along *AB* be F_A, the force acting along *CB* be F_C, and the weight of *P* be F_p. The horizontal component of F_A is $F_A \sin\angle ABG$, opposing the horizontal component of F_C, which is $F_C \sin\angle CBG$; and F_p is the force of gravity on *P*, and only acts vertically. Because *P* is not moving horizontally,

$$F_A \sin\angle ABG - F_C \sin\angle CBG = 0,$$

or

$$F_A \sin\angle ABG = F_C \sin\angle CBG,$$

and so

$$F_A : F_C = \sin\angle CBG : \sin\angle ABG,$$

as stated in point 5 of this lecture. The vertical component of F_A is $F_A \cos\angle ABG$, pulling upward; the vertical component of F_C is $F_C \cos\angle CBG$, also pulling upward; and F_p, the weight of *P*, is of course pulling down. Because *P* is not moving vertically, we have

$$F_p - (F_A \cos\angle ABG + F_C \cos\angle CBG) = 0,$$

or

$$F_p = F_A \cos\angle ABG + F_C \cos\angle CBG.$$

Dividing both sides of the equation by F_C yields

$$\frac{F_p}{F_C} = \frac{F_A}{F_C}\cos\angle ABG + \cos\angle CBG.$$

We proved above that $F_A/F_C = \sin\angle CBG / \sin\angle ABG$, so we can substitute here and obtain

$$\frac{F_p}{F_C} = \frac{\sin\angle CBG \cos\angle ABG}{\sin\angle ABG} + \cos\angle CBG,$$

and multiplying $\cos\angle CBG$ by $1 = \sin\angle ABG / \sin\angle ABG$, and rearranging terms in the equation, we get

$$\frac{F_p}{F_C} = \frac{\sin\angle CBG \cos\angle ABG + \cos\angle CBG \sin\angle ABG}{\sin\angle ABG}.$$

Now $\angle CBG + \angle ABG = \angle ABC$, and the Greeks knew that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

so we have

$$\sin\angle ABC = \sin\angle CBG \cos\angle ABG + \cos\angle CBG \sin\angle ABG,$$

and substituting,

$$\frac{F_p}{F_C} = \frac{\sin\angle ABC}{\sin\angle ABG},$$

or $F_p : F_C = \sin\angle ABC : \sin\angle ABG$, as stated in point 5. [Ferguson]

3. The force at *B* is a constant, expressible as the weight of a segment of chain labelled *C,* of constant length *a* (not related to point *a*). [Ferguson]

4. If the chain is of uniform density per unit length (call the density *Q*), then the weight of chain *AB* will be *Qs,* which increases or decreases as the chain *AB = s* is lengthened or shortened; the force at *B,* which Bernoulli sets equal to the weight of a segment *C* of constant length *a,* will be *Qa.* Therefore, the weight of *AB* is to the force at *B* as *Qs* is to *Qa,* and therefore as *s* is to *a.* Bernoulli expresses this ratio of forces in terms of a ratio of lengths, so that it will be in a form solvable by the method of inverse tangents, as is shown below. [Ferguson]

5. Here *y* is considered an independent variable (instead of *x*). Accordingly, $d^2y = 0$ This aid for the conversion of independent variables is already found in Leibniz manuscripts of 1675.

   Bernoulli cannot complete the calculation, because he lacks the logarithmic function. From

$$dy = \frac{a\,dx}{\sqrt{x^2 - a^2}}$$

it follows that

$$y = a \log\left(x + \sqrt{x^2 - a^2}\right) + \text{const.}$$

or, if the axis system is displaced appropriately in the *y*-direction,

$$y = a \log \frac{x + \sqrt{x^2 - a^2}}{a}.$$

From this it follows that

$$x = \frac{a}{2}\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right).$$

The curve is therefore the catenary, which Bernoulli deals with in a later lecture. [Kowalewski]

6. For the equilateral hyperbola *BC,* $(x + a)^2 - y^2 = a^2$ or $y = \sqrt{2ax + x^2}$. Therefore $KC = \sqrt{2ax + x^2}$. If $BA = a$ is the mean proportional between *KC* and *KD,* then $KC/a = a/KD$ or $KD = a^2/\sqrt{2ax + x^2}$. Then, if the area of rectangle $AG = aAF$ is to equal the area of

$$HBKDJ = \int \frac{a^2\,dx}{\sqrt{2ax + x^2}},$$

then for the curve *BE,*

$$y = EK = AF = \int \frac{a\,dx}{\sqrt{2ax + x^2}},$$

and

$$dy = \frac{a\,dx}{\sqrt{2ax + x^2}}.$$

[Ferguson]

7. For the equilateral hyperbola,

$$\frac{dx}{y} = \frac{dy}{x} = \frac{x\,dy - y\,dx}{x^2 - y^2} = \frac{dx + dy}{x + y},$$

therefore

$$\frac{1}{2}(x\,dy - y\,dx) = \frac{a^2}{2}\,d\log(x + y)$$

and the hyperbolic sector

$$ABC = \frac{a^2}{2}\log\frac{x + \sqrt{x^2 - a^2}}{a}.$$

If this be compared with the formula in footnote 5,

$$y = a\log\frac{x + \sqrt{x^2 - a^2}}{a},$$

the correctness of Bernoulli's assertion is seen.

On page 37, col. 1, line 23, *A* must be replaced by *B.* [Kowalewski] [For this translation, the correction has been incorporated into the text.]

8. If the area of rectangle *AG* is equal to double the area of hyperbolic region *ABC,* then we have $aAF = 2(\text{area } ABC)$ or $y = EK = AF$

$$= \frac{2(\text{area } ABC)}{a} = \frac{2}{a}\left(\frac{1}{2}\,AK \cdot KC - \text{area } BKC\right)$$

$$= \frac{2}{a}\left[\frac{1}{2}(x + a)\sqrt{2ax + x^2} - \int\sqrt{2ax + x^2}\,dx\right],$$

and $dy = \dfrac{a\,dx}{\sqrt{2ax + x^2}}.$

[Ferguson]

9. The German edition has *AE* here, an error for *BE.* The expression, "the other endpoint *E* again will lie on the desired curve," taken in light of the two preceding paragraphs, indicates that *BE* is meant. This is, in fact, the same error that Kowalewski corrected on the previous page of the German edition: The vertex of the catenary had been mistakenly called *A* instead of *B.* See the last paragraph of footnote 7. [Ferguson]

10. "Hence one must seek a certain curve *BL*" whose *length s* as a function of *x,* has the differential

$$ds = \frac{dx\sqrt{2a + x}}{\sqrt{x}}.$$

Since $ds^2 = dx^2 + dy^2,$

$$dy^2 = ds^2 - dx^2 = \frac{(2a + x)dx^2}{x} - dx^2 = \frac{2a\,dx^2}{x}$$

and

$$dy = \frac{dx\sqrt{2a}}{\sqrt{x}} = \text{ the differential of } KL,$$

that is, the *y*-coordinate of *BL.* So for *BL,*

$$y = \sqrt{8ax}.$$

[Ferguson]

11. The logarithmic curve is, because of the constancy of the subtangent, characterized by $y' = y/c$ [that is, $dy/dx = y/c$] and is therefore expressed by $y = be^{cx}.$

Because of its relationship with the logarithmic curve, Leibniz proposed to use the catenary in place of logarithmic tables, especially when travelling, because voluminous tables were inconvenient "to *schlep* across land and sea." [Kowalewski]

12. Definition of the subtangent: For a curve $y = f(x)$ with a line *L* tangent to it at point (x',y') the subtangent is the segment of the *x*-axis between the *x*-coordinate of the point of tangency (x',o) and the intersection of the *x*-axis and the tangent line *L.* [Ferguson]

13. If the length of $BE = s$ is considered a function of *x,* then

$$BE = s = \int ds = \int\sqrt{dx^2 + dy^2} = \int\sqrt{dx^2 + \frac{a^2\,dx^2}{2ax + x^2}},$$

which reduces to

$$\int \frac{(x + a)dx}{\sqrt{2ax + x^2}} \quad \text{or} \quad \sqrt{2ax + x^2},$$

which is the *y*-coordinate *AG* of the equilateral hyperbola *BG.* [Ferguson]

14. Because

$$Ee = ds = \frac{(x + a)dx}{\sqrt{2ax + x^2}} \quad \text{and} \quad ER = dy = \frac{a\,dx}{\sqrt{2ax + x^2}}.$$

[Ferguson]

15. The proposed constructions may be restated in this form: Suppose the catenary *EBF,* in Figure 9, to be a chain with very fine links. Step 1. At various points along *EBF,* cut rods that extend vertically from each chosen point on the catenary curve to the horizontal *EF.* Attach each rod to the link it touches. Now extend the chain as a straight line. Use the free ends of the still-vertical rods to define a line. An area is now enclosed by the straight chain and the line just defined, and can be measured. Step 2. Form any number of other curves whose end-points are *E* and *F,* and whose length is the same as the catenary *EBF.* Repeat for these curves the procedure described in Step 1.

Bernoulli asserts that the area derived from the catenary is the greatest of all such areas. [Ferguson]